Tindall et al

$$\frac{\partial b}{\partial t} = \nabla \cdot (\mu(s)\nabla b) - \nabla \cdot (\chi(s) b \nabla s)$$
$$+ g(b,s) - h(b,s)$$

$$\frac{\partial s}{\partial t} = D\nabla^2 s - f(b,s)$$

Where

b - bacterial density

s - attractant conc$^{\underline{n}}$

μ - bacterial diff$^{\underline{n}}$ coeff

χ - chemotactic coeff

g - cell growth

h - cell death

D - diffusion coeff of chemoattractant

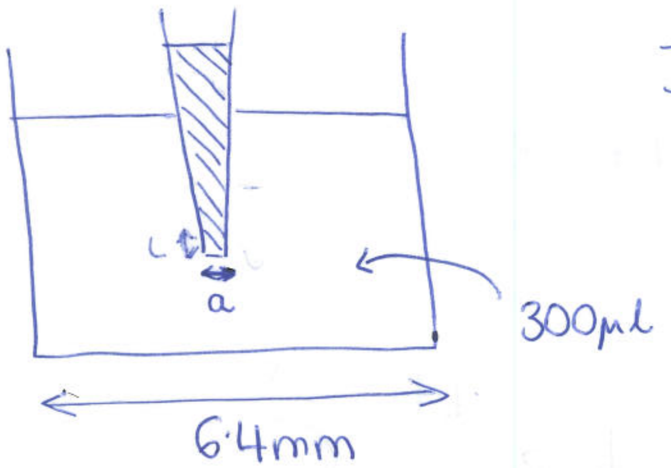
f - attractant degradation

Simplified

$$\frac{\partial b}{\partial t} = \mu \nabla^2 b - \chi \nabla \cdot (b \nabla s)$$

$$\frac{\partial s}{\partial t} = D\nabla^2 s$$

Chemoattractant problem



Initially

In pipette : $c = c_0$, $b = 0$

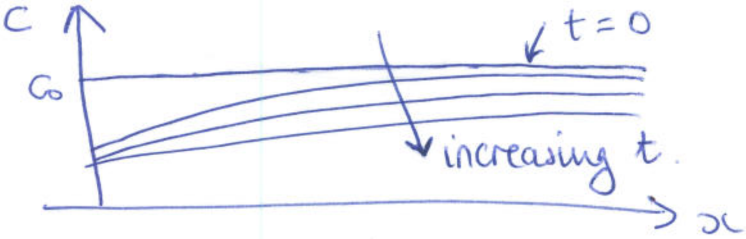
In bulk : $c = 0$, $b = b_0$

We have
$$\frac{\partial b}{\partial t} = \mu \nabla^2 b - \chi \underline{\nabla} \cdot (b \underline{\nabla} c)$$
$$\frac{\partial c}{\partial t} = D \nabla^2 c$$

Also $\underline{n} \cdot \underline{\nabla} b = \underline{n} \cdot \underline{\nabla} c = 0$ on all surfaces

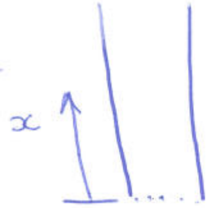
For simplicity we assume the pipette is a cylinder.

The concentration in the pipette is expected to have the form



The lengthscale of propagation of loss of concentration up the tube is given by
$$L = \sqrt{Dt}$$
Assuming $D = 10^{-10}\,m^2s$ then after an hour, the lengthscale is 0.6 mm, which we assume to be small compared to the height of fluid in the pipette. Thus we neglect concentration loss in the pipette and assume $c = c_0$

Pipette



In (P) (Pipette)

$$\frac{\partial c_p}{\partial t} = D \frac{\partial^2 c_p}{\partial x^2} \qquad c_p = c_0 \text{ at } t=0$$

Laplace transform

$$C_p = \int_0^\infty e^{-st} c_p \, dt$$

$$\Rightarrow s C_p - c_0 = D \frac{\partial^2 C_p}{\partial x^2}$$

Solution

$$C_p = \frac{c_0}{s} + A_p(s) e^{-\sqrt{\frac{s}{D}}x} + B_p(s) e^{+\sqrt{\frac{s}{D}}x}$$

Assume Surface of fluid is far away (pretend it is at ∞)

$$\Rightarrow C_p \to \frac{c_0}{s} \quad \text{as } x \to \infty$$

$$\therefore B_p = 0$$

So $$C_p = \frac{c_0}{s} + A_p(s) e^{-\sqrt{\frac{s}{D}}x}$$

Bulk

(distance from tip of pipette)

In (B) (bulk)

$$\frac{\partial c_B}{\partial t} = D \frac{1}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial c_B}{\partial r}\right)$$

(assume spherical symmetry)

Laplace transform

$$C_B = \int_0^\infty e^{-st} c_B \, dt$$

$$\Rightarrow s C_B = \frac{D}{r^2} \frac{\partial}{\partial r}\left(r^2 \frac{\partial C_B}{\partial r}\right)$$

Solution $$C_B = \frac{1}{r}\left(A_B(s) e^{-\sqrt{\frac{s}{D}}r} + B_B(s) e^{+\sqrt{\frac{s}{D}}r}\right)$$

Assume boundaries of container are far away (at ∞) $\therefore$ $C_B \to 0$ as $r \to \infty$

$\therefore B_B = 0.$

So $$C_B = \frac{A_B(s)}{r} e^{-\sqrt{\frac{s}{D}}\, r}$$

<u>Matching</u> We model the surface of the pipette in the bulk as a sphere having the same surface area as the real surface.

a = diameter of pipette

R = radius of sphere.

$$4\pi R^2 = \frac{\pi a^2}{4} \Rightarrow R = \frac{a}{4}$$

So
$$\left.\begin{aligned} c_p\big|_{x=0} &= c_B\big|_{r=R} \\ \frac{\pi a^2}{4} D \frac{\partial c_p}{\partial x}\bigg|_{x=0} &= -4\pi R^2 D \frac{\partial c_B}{\partial r}\bigg|_{r=R} \end{aligned}\right\}$$
(in terms of original variables)

$$\Rightarrow \left.\begin{aligned} C_p\big|_{x=0} &= C_B\big|_{r=R} \\ \frac{\partial C_p}{\partial x}\bigg|_{x=0} + \frac{\partial C_B}{\partial r}\bigg|_{r=R} &= 0 \end{aligned}\right\}$$
(in terms of Laplace transform variables)

Therefore:
$$\begin{cases} \dfrac{C_0}{s} + A_p(s)e^0 = \dfrac{A_B(s)}{R} e^{-\sqrt{\frac{s}{D}}R} \\ -A_p(s)\sqrt{\dfrac{s}{D}} + A_B(s)\left(-\dfrac{1}{R^2} - \dfrac{1}{R}\sqrt{\dfrac{s}{D}}\right)e^{-\sqrt{\frac{s}{D}}R} = 0 \end{cases}$$

$$A_B = \frac{R^2 \cdot c_0 e^{\sqrt{\frac{s}{D}} R}}{s\left(\sqrt{\frac{D}{s}} + 2R\right)}$$

$$A_p = -\frac{c_0\left(\sqrt{\frac{D}{s}} + R\right)}{s\left(\sqrt{\frac{D}{s}} + 2R\right)}$$

Hence
$$\begin{cases} C_p = \dfrac{c_0}{s}\left(1 \overset{-}{\cancel{+}} \dfrac{\sqrt{\frac{D}{s}} + R}{\sqrt{\frac{D}{s}} + 2R} e^{-\sqrt{\frac{s}{D}} x}\right) \\ C_B = \dfrac{R^2 c_0 e^{\sqrt{\frac{s}{D}}(R-r)}}{r s\left(\sqrt{\frac{D}{s}} + 2R\right)} \end{cases}$$

Abramowitz & Stegun (Dover, 1972) A Handbook of Mathematical Functions states,

(29.3.90) p. 1027

$$\mathcal{L}^{-1}\left[\frac{e^{-k\sqrt{s}}}{\sqrt{s}(a+\sqrt{s})}\right](t) = e^{ak} e^{a^2 t} \operatorname{erfc}\left(a\sqrt{t} + \frac{k}{2\sqrt{t}}\right) \quad (k \geq 0)$$

(29.3.83) p. 1026 $\quad \mathcal{L}^{-1}\left[\frac{1}{s} e^{-k\sqrt{s}}\right](t) = \operatorname{erfc}\left(\frac{k}{2\sqrt{t}}\right) \quad (k \geq 0)$

Hence $C_p = \mathcal{L}^{-1}\left[\frac{c_0}{s}\left(1 \overset{-}{\cancel{+}} \frac{\sqrt{\frac{D}{s}} + R}{\sqrt{\frac{D}{s}} + 2R} e^{-\sqrt{\frac{s}{D}} x}\right)\right](t)$

$$= \mathcal{L}^{-1}\left[\frac{c_0}{s}\right](t) \overset{-}{\cancel{+}} \mathcal{L}^{-1}\left[\frac{c_0}{s} e^{-\sqrt{\frac{s}{D}} x}\right](t) + \mathcal{L}^{-1}\left[+\frac{c_0}{s} \frac{R e^{-\sqrt{\frac{s}{D}} x}}{\sqrt{\frac{D}{s}} + 2R}\right](t)$$

$$= c_0 \overset{-}{\cancel{+}} c_0 \operatorname{erfc}\left(\frac{x}{2\sqrt{Dt}}\right) + \frac{c_0}{2} e^{x/2R} e^{\frac{Dt}{4R^2}} \operatorname{erfc}\left(\frac{\sqrt{Dt}}{2R} + \frac{x}{2\sqrt{Dt}}\right)$$

$$C_B = \mathcal{L}^{-1}\left[\frac{R^2 c_0 e^{\sqrt{\frac{s}{D}}(R-r)}}{r s\left(\sqrt{\frac{D}{s}} + 2R\right)}\right](t)$$

$$= \frac{R c_0}{2r} e^{\frac{r-R}{2R}} e^{\frac{Dt}{4R^2}} \operatorname{erfc}\left(\frac{\sqrt{Dt}}{2R} + \frac{r-R}{2\sqrt{Dt}}\right)$$

$$\text{So} \quad c = \begin{cases} c_0 \left\{ \operatorname{erf}\left(\frac{x}{2\sqrt{Dt}}\right) + \frac{1}{2} e^{x/2R} e^{Dt/4R^2} \operatorname{erfc}\left(\frac{\sqrt{Dt}}{2R} + \frac{x}{2\sqrt{Dt}}\right) \right\} & \text{in pipette} \\ \frac{Rc_0}{2r} e^{\frac{r-R}{2R}} e^{\frac{Dt}{4R^2}} \operatorname{erfc}\left(\frac{\sqrt{Dt}}{2R} + \frac{r-R}{2\sqrt{Dt}}\right) & \text{in bulk} \end{cases}$$

__Bacterial concentration__

If $x \gg \sqrt{Dt}$, $r \gg R + 2\sqrt{Dt}$ $\left(\cancel{t \ll \frac{4R^2}{D}}\right)$

$$c \approx \begin{cases} c_0 & \text{in pipette} \\ 0 & \text{in bulk} \end{cases}$$

In these regions $\frac{\partial b}{\partial t} = \mu \nabla^2 b$

$\therefore$ lengths scale as $\sqrt{\mu t}$