Preparation

Considering the Hill equation in the simplification DDEs, $A1_{c1}$ and K_{M1}/ρ_1 should be the same order of magnitude, thus K_{M1}/ρ_1 is a well measurement of quantities of $A1_{c1}$. We have:

$$[A1_{c1}] \sim K_{M1}/\rho_1$$

Similarly,

$$[A2_{C2}] \sim K_{M4}/\rho_2$$

In equation (20) (23), Let:

$$dA1_{c1}/dt = 0, dA2_{c2}/dt = 0$$

We have:

$$[A1_e] \sim \frac{(k_a + \gamma) \cdot K_{M1}}{\gamma \cdot \rho_1}, \quad [A2_e] \sim \frac{(k_b + \gamma) \cdot K_{M4}}{\gamma \cdot \rho_2}$$

In equation (24) (25), Let:

$$dA1_e/dt = 0, dA2_e/dt = 0$$

We have:

$$[A1_{c2}] \sim \frac{K_{M1}}{\rho_1} \cdot (1 + \frac{\mu}{\gamma} \cdot \frac{1 - p \cdot (1 + n_{12})}{p})$$

$$[A2_{c1}] \sim \frac{K_{M4}}{\rho_2} \cdot (1 + \frac{\mu}{\gamma} \cdot \frac{1 - p \cdot (1 + n_{12})}{p \cdot n_{12}})$$

Define:

$$x_1 = \frac{A1_{c1}}{K_{M1}/\rho_1}, \qquad x_2 = \frac{A1_{c2}}{K_{M1}/\rho_1} \cdot \frac{1}{1 + \frac{\mu}{\gamma} \cdot \frac{1 - p(1 + n_{12})}{p}},$$

$$y_1 = \frac{A2_{c1}}{K_{M4}/\rho_2} \cdot \frac{1}{1 + \frac{\mu}{\gamma} \cdot \frac{1 - p(1 + n_{12})}{p \cdot n_{12}}}, \qquad y_2 = \frac{A2_{C2}}{K_{M4}/\rho_2},$$

$$x_e = \frac{A1_e}{K_{M1}/\rho_1} \cdot \frac{k_a + \gamma}{\gamma}, \qquad y_e = \frac{A2_e}{K_{M4}/\rho_2} \cdot \frac{k_b + \gamma}{\gamma}, \qquad t^* = \gamma \cdot t$$

Define:

$$a = \frac{k_a}{\gamma}, \quad b = \frac{k_b}{\gamma}, \quad \mathrm{u} = \frac{\mu}{\gamma}, \quad \mathrm{v} = \frac{p}{1 - p(1 + \mathrm{n}_{12})}, \quad \mathrm{m} = \frac{\rho_1}{\mathrm{K}_{\mathrm{M1}}} \cdot \frac{k_{p2}}{r}, \quad \mathrm{n} = \frac{\rho_2}{\mathrm{K}_{\mathrm{M4}}} \cdot \frac{k_{p1}}{r}$$

The dimensionless equations are as follows:

$$\frac{dx_1}{dt^*} = -(a+1) \cdot x_1 + (1+a) \cdot x_e \tag{26}$$

$$\frac{dy_2}{dt^*} = -(b+1) \cdot y_2 + (1+b) \cdot y_e \tag{27}$$

$$\frac{dx_2}{dt^*} = m \frac{1}{1 + u/\mathrm{v}} \cdot \frac{1}{1 + y_2(t^* - \tau_1{}^*)^{n2}} + \frac{1}{1 + u/\mathrm{v}} \cdot \frac{1}{a+1} x_e - (a+1) x_2 \tag{28}$$

$$\frac{dy_1}{dt^*} = n \frac{1}{1 + u/\mathrm{vn}_{12}} \cdot \frac{x_1(t^* - \tau_2{}^*)^{n1}}{1 + x_1(t^* - \tau_2{}^*)^{n1}} + \frac{1}{1 + u/\mathrm{vn}_{12}} \cdot \frac{1}{b+1} y_e - (b+1) y_1 \tag{29}$$

$$\frac{dx_e}{dt^*} = -(u + (1 + \mathrm{n}_{12}) \cdot v) \cdot x_e + (1+a) \cdot v\mathrm{n}_{12} \cdot x_1 + (1+a) \cdot v \cdot \left(1 + \frac{\mathrm{u}}{\mathrm{v}}\right) x_2 \tag{30}$$

$$\frac{dy_e}{dt^*} = -(u + (1 + \mathrm{n}_{12}) \cdot v) \cdot y_e + (1+b) \cdot v \cdot y_2 + (1+b) \cdot v\mathrm{n}_{12} \cdot \left(1 + \frac{\mathrm{u}}{\mathrm{vn}_{12}}\right) y_1 \tag{31}$$